Maxwell's field equation in covariant (Invariant) tensor form

Maxwell's field equation in differential form,

$\vec{\nabla}\cdot\vec{D} = \rho$ ——— (i) $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$ ——— (ii)

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ ——— (iii) $\qquad$ $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ ——— (iv)

where electric displacement $\vec{D} = \varepsilon_0\varepsilon_r\vec{E}$

and magnetic field $\vec{B} = \mu_0\mu_r\cdot\vec{H}$

For simplicity, $\varepsilon_r = 1$ and $\mu_r = 1$ (for free space)

Using these in above equations, We get

$$\vec{\nabla}\cdot\varepsilon_0\vec{E} = \rho \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{——— ①}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0 \quad \text{——— ②}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \quad \text{——— ③}$$

$$\vec{\nabla}\times\frac{\vec{B}}{\mu_0} = \vec{J} + \frac{\partial\varepsilon_0\vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

$$\Rightarrow \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \quad \text{——— ④}$$

Now introducing the coordinates- $\qquad \because c^2 = \frac{1}{\mu_0\varepsilon_0}$.

$x = x_1$, $y = x_2$, $z = x_3$ and $ict = x_4$.

The above equations can be written as

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{——— ⑤}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{——— ⑥}$$

$$\vec{\nabla}\times\vec{E} + ic\frac{\partial\vec{B}}{\partial(ict)} = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E} + ic\frac{\partial\vec{B}}{\partial x_4} = 0 \quad \text{——— ⑦} \qquad \because ict = x_4$$

and

$$\vec{\nabla}\times\vec{B} - \frac{i}{c}\cdot\frac{\partial\vec{E}}{\partial(ict)} = \mu_0\vec{J}$$

$$\Rightarrow \vec{\nabla}\times\vec{B} - \frac{i}{c}\cdot\frac{\partial\vec{E}}{\partial x_4} = \mu_0\vec{J} \quad \text{——— ⑧} \qquad \because ict = x_4$$

We have introduced fourth component x_4 of displacement four vector in the Maxwell's equation.

From equ^n (8), $\vec{\nabla}\times\vec{B} - \frac{i}{c}\cdot\frac{\partial \vec{E}}{\partial x_4} = \mu_0 \vec{J}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ B_1 & B_2 & B_3 \end{vmatrix} - \frac{i}{c}\frac{\partial (E_1\hat{i}+E_2\hat{j}+E_3\hat{k})}{\partial x_4} = \mu_0 (J_1\hat{i}+J_2\hat{j}+J_3\hat{k})$$

$$\Rightarrow \hat{i}\left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}\right) - \hat{j}\left(\frac{\partial B_3}{\partial x_1} - \frac{\partial B_1}{\partial x_3}\right) + \hat{k}\left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2}\right)$$
$$- \frac{i}{c}\frac{\partial}{\partial x_4}(E_1\hat{i}+E_2\hat{j}+E_3\hat{k}) = \mu_0 (J_1\hat{i}+J_2\hat{j}+J_3\hat{k}) \quad \text{—— (9)}$$

Equating components (x, y & z) from equ^n (9), we get

$$0 + \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{i}{c}\cdot\frac{\partial E_1}{\partial x_4} = \mu_0 J_1 \quad \text{—— (10)}$$

$$-\frac{\partial B_3}{\partial x_1} + 0 + \frac{\partial B_1}{\partial x_3} - \frac{i}{c}\frac{\partial E_2}{\partial x_4} = \mu_0 J_2 \quad \text{—— (11)}$$

$$\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} + 0 - \frac{i}{c}\frac{\partial E_3}{\partial x_4} = \mu_0 J_3 \quad \text{—— (12)}$$

From equ^n (5), $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$$\Rightarrow \left(\frac{\partial}{\partial x_1}\hat{i} + \frac{\partial}{\partial x_2}\hat{j} + \frac{\partial}{\partial x_3}\hat{k}\right)\cdot(E_1\hat{i}+E_2\hat{j}+E_3\hat{k}) = \frac{ic\rho}{ic\epsilon_0}$$

$$\Rightarrow \frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = \frac{J_4}{ic\epsilon_0} \qquad \because ic\rho = J_4$$

Multiplying bothsides by $\frac{i}{c}$, we get

$$\Rightarrow \frac{i}{c}\cdot\frac{\partial E_1}{\partial x_1} + \frac{i}{c}\frac{\partial E_2}{\partial x_2} + \frac{i}{c}\frac{\partial E_3}{\partial x_3} = \frac{J_4}{c^2\epsilon_0}$$

$$\Rightarrow \frac{i}{c}\cdot\frac{\partial E_1}{\partial x_1} + \frac{i}{c}\cdot\frac{\partial E_2}{\partial x_2} + \frac{i}{c}\cdot\frac{\partial E_3}{\partial x_3} + 0 = \mu_0 J_4 \quad \text{—— (13)}$$

$$\because c^2 = \frac{1}{\mu_0\epsilon_0}$$

Treating the right hand members of the system of equ^{n}s (10 - 13) as the components of current four vector (J_μ) and introducing in the LHS, members as a set of dependent variables defined by $F_{\mu\nu}$ (EM field tensor)

From LHS of eqⁿ ⑩, $F_{11}=0$, $F_{12}=B_3$, $F_{13}=-B_2$, $F_{14}=-\frac{i}{c}E_1$

From LHS of eqⁿ ⑪, $F_{21}=-B_3$, $F_{22}=0$, $F_{23}=B_1$, $F_{24}=-\frac{i}{c}E_2$

From LHS of eqⁿ ⑫, $F_{31}=B_2$, $F_{32}=-B_1$, $F_{33}=0$, $F_{34}=-\frac{i}{c}E_3$

From LHS of eqⁿ ⑬, $F_{41}=\frac{i}{c}E_1$, $F_{42}=\frac{i}{c}E_2$, $F_{43}=\frac{i}{c}E_3$, $F_{44}=0$

So

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

$$\Rightarrow F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{bmatrix} \quad \text{—— Ⓐ}$$

In compact form, eqⁿˢ ⑩ to ⑬ or eqⁿˢ ① & ④ or eqⁿˢ ⑤ & ⑧ can be written in single equation as

$$\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu \quad \text{—— Ⓑ}$$

where $\mu = 1, 2, 3$ & 4.

Illustration

(a) If $\mu=1$ then eqⁿ Ⓑ will become

$$\sum_{\nu=1}^{4} \frac{\partial F_{1\nu}}{\partial x_\nu} = \mu_0 J_1$$

$$\Rightarrow \frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_2} + \frac{\partial F_{13}}{\partial x_3} + \frac{\partial F_{14}}{\partial x_4} = \mu_0 J_1$$

$$\Rightarrow \frac{\partial 0}{\partial x_1} + \frac{\partial B_3}{\partial x_2} + \frac{\partial(-B_2)}{\partial x_3} + \frac{\partial(-\frac{i}{c}E_1)}{\partial x_4} = \mu_0 J_1$$

$$\Rightarrow 0 + \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{i}{c}\frac{\partial E_1}{\partial x_4} = \mu_0 J_1$$ It is eqⁿ ⑩

(b) If $\mu = 4$ then equ^n (B) will become

$$\sum_{\nu=1}^{4} \frac{\partial F_{4\nu}}{\partial x_\nu} = \mu_0 J_4$$

$$\Rightarrow \frac{\partial F_{41}}{\partial x_1} + \frac{\partial F_{42}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_3} + \frac{\partial F_{44}}{\partial x_4} = \mu_0 J_4$$

$$\Rightarrow \frac{\partial}{\partial x_1}\left(\frac{i}{c}E_1\right) + \frac{\partial}{\partial x_2}\left(\frac{i}{c}E_2\right) + \frac{\partial}{\partial x_3}\left(\frac{i}{c}E_3\right) + \frac{\partial}{\partial x_4}(0) = \mu_0 J_4$$

$$\Rightarrow \frac{i}{c}\frac{\partial E_1}{\partial x_1} + \frac{i}{c}\frac{\partial E_2}{\partial x_2} + \frac{i}{c}\cdot\frac{\partial E_3}{\partial x_3} + 0 = \mu_0 J_4 \quad \text{It is equ}^n \text{ (13)}.$$

Now from equ^n (7), $\vec{\nabla} \times \vec{E} + ic\frac{\partial \vec{B}}{\partial x_4} = 0$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ E_1 & E_2 & E_3 \end{vmatrix} + ic\frac{\partial}{\partial x_4}(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) = 0$$

$$\Rightarrow \hat{i}\left(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}\right) - \hat{j}\left(\frac{\partial E_3}{\partial x_1} - \frac{\partial E_1}{\partial x_3}\right) + \hat{k}\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right)$$
$$+ ic\frac{\partial B_1}{\partial x_4}\hat{i} + ic\frac{\partial B_2}{\partial x_4}\hat{j} + ic\frac{\partial B_3}{\partial x_4}\hat{k} = 0 \quad \text{--- (14)}$$

Writing equ^n (13) in components form, we get

$$0 + \frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} + ic\frac{\partial B_1}{\partial x_4} = 0 \quad \text{--- (15)}$$

$$-\frac{\partial E_3}{\partial x_1} + 0 + \frac{\partial E_1}{\partial x_3} + ic\frac{\partial B_2}{\partial x_4} = 0 \quad \text{--- (16)}$$

$$\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} + 0 + ic\frac{\partial B_3}{\partial x_4} = 0 \quad \text{--- (17)}$$

Multiplying both sides of equ^n (15), (16) & (17) by $\frac{1}{ic} = -\frac{i}{c}$, we get

$$0 - \frac{i}{c}\frac{\partial E_3}{\partial x_2} + \frac{i}{c}\frac{\partial E_2}{\partial x_3} + \frac{\partial B_1}{\partial x_4} = 0 \quad \text{--- (18)}$$

$$\frac{i}{c}\frac{\partial E_3}{\partial x_1} + 0 - \frac{i}{c}\frac{\partial E_1}{\partial x_3} + \frac{\partial B_2}{\partial x_4} = 0 \quad \text{--- (19)}$$

$$-\frac{i}{c}\frac{\partial E_2}{\partial x_1} + \frac{i}{c}\cdot\frac{\partial E_1}{\partial x_2} + 0 + \frac{\partial B_3}{\partial x_4} = 0 \quad \text{--- (20)}$$

From equⁿ (6), $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \left(\hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}\right)\cdot(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) = 0$

$$\Rightarrow \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0$$

$$\Rightarrow \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} + 0 = 0 \quad \text{———} \quad (21)$$

By using electromagnetic field tensor $F_{\mu\nu}$

where
$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = \begin{bmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{bmatrix}$$

Equⁿs (18) to (21) can be written as

$$0 + \frac{\partial F_{34}}{\partial x_2} + \frac{\partial F_{42}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_4} = 0 \quad \text{———} \quad (22)$$

$$\frac{\partial F_{43}}{\partial x_1} + 0 + \frac{\partial F_{14}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_4} = 0 \quad \text{———} \quad (23)$$

$$\frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{41}}{\partial x_2} + 0 + \frac{\partial F_{12}}{\partial x_4} = 0 \quad \text{———} \quad (24)$$

$$\frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_3} + 0 = 0 \quad \text{———} \quad (25)$$

All these equⁿs from (22) to (25) can be written more compactly in single equation as

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \quad \text{———} \quad (C)$$

It represents Maxwell's equⁿs (6) & (7) or equⁿs (2) & (3).

Illustration:

(a) If $\lambda = 1$, $\mu = 2$ and $\nu = 3$ then equⁿ (C) will become

$$\frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} = 0 \Rightarrow \frac{\partial B_3}{\partial x_3} + \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} = 0$$ It is equⁿ (21)

or $\vec{\nabla}\cdot\vec{B} = 0$ → Maxwell's equation.

(b) If $\lambda=1$, $\mu=2$ and $\nu=4$ then equ ⓒ will become

$$\frac{\partial F_{12}}{\partial x_4}+\frac{\partial F_{24}}{\partial x_1}+\frac{\partial F_{41}}{\partial x_2}=0 \Rightarrow \frac{\partial B_3}{\partial x_4}+\frac{\partial(-\frac{i}{c}E_2)}{\partial x_1}+\frac{\partial(\frac{i}{c}E_1)}{\partial x_2}=0$$

$$\Rightarrow \frac{\partial B_3}{\partial x_4}-\frac{i}{c}\frac{\partial E_2}{\partial x_1}+\frac{i}{c}\frac{\partial E_1}{\partial x_2}=0$$

Multiplying bothsides by $-\frac{c}{i}=ic$, we get

$$\Rightarrow ic\frac{\partial B_3}{\partial x_4}+\frac{\partial E_2}{\partial x_1}-\frac{\partial E_1}{\partial x_2}=0$$

$$\Rightarrow \frac{\partial E_2}{\partial x_1}-\frac{\partial E_1}{\partial x_2}+ic\frac{\partial B_3}{\partial x_4}=0$$

$$\Rightarrow (\vec{\nabla}\times\vec{E})_3+ic\frac{\partial B_3}{\partial x_4}=0 \text{ or } (\vec{\nabla}\times\vec{E})_3+\frac{\partial B_3}{\partial t}=0 \quad \text{It is (17)}$$

or z-components of Maxwell's equ ③ or ⑦

Therefore Maxwell's field equation can be written in terms of electromagnetic field tensor $F_{\mu\nu}$.

Maxwell's field equation in terms of electromagnetic field tensor $F_{\mu\nu}$ are

$$\sum_{\nu=1}^{4}\frac{\partial F_{\mu\nu}}{\partial x_\nu}=\mu_0 J_\mu \quad \text{———— (B)}$$

$$\& \quad \frac{\partial F_{\lambda\mu}}{\partial x_\nu}+\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}=0 \quad \text{———— (C)}$$

μ_0 = Scalar and J_μ = current four vector product $\mu_0 J_\mu$ should be four vector. Divergence of any tensor of rank 2 will be vector.

$\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ = divergence of EM field tensor $F_{\mu\nu}$

so $F_{\mu\nu}$ should be a tensor of rank 2.

Equ (B) is tensor equ of rank-1 but equ (C) is tensor equ of rank-3.

* All equations in tensor form are invariant or covariant under Lorentz (coordinate) transformation.

Therefore equ's (B) and (C) are simply known as covariant (Invariant) 4 tensor form of Maxwell's equation.

Note :- If we have to prove invariant of any equation then if we express the equation in tensor form then the equation will be invariant under Lorentz transformation. —×—